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INVARIANT SOLUTIONS OF THE THERMAL-DIFFUSION EQUATIONS FOR A BINARY MIXTURE IN THE CASE OF PLANE MOTION

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The group properties of the thermal-diffusion equations for a binary mixture in plane flow are studied. Optimal systems of first-and second-order subalgebras are constructed for the admissible Lie operator algebra, which is infinite-dimensional. Examples of the exact invariant solutions are given, which are found by solving ordinary differential equations. Exact solutions are found that describe thermal diffusion in an inclined layer with a free boundary and in a vertical layer in the presence of longitudinal temperature and concentration gradients. The effect of the thermal-diffusion parameter on the flow regime is studied.

Key words: *thermal diffusion, binary mixture, group analysis, invariant solutions.*

Introduction. Thermal diffusion is molecular transfer of material due to the presence of a temperature gradient in the medium (liquid solution or gas mixture). In the case of thermal diffusion, the components have different concentrations in the regions of elevated and decreased temperature. The presence of concentration gradients results in ordinary diffusion. A steady state is established when the diffusion and thermal-diffusion processes compensate for each other (i.e., the process of separation of the mixture components is compensated by the process of their mixing). In practice, a frequently occurring case is normal thermal diffusion, in which the heavier components move to the colder regions and the lighter components pass to the more heated regions. In some cases, there may be anomalous thermal diffusion, in which the direction of motion of the components is opposite.

The present paper considers a model for the convective motion of a binary mixture taking into account thermal diffusion. The model is based on the Navier–Stokes equations supplemented by diffusion and heat-transfer equations. The Oberbeck–Boussinesq approximation, intended to describe convective flows under natural earth conditions, is used. It is assumed that the density of the mixture depends linearly on the temperature and concentration of the light component: $\rho = \rho_0(1 - \beta_1 T - \beta_2 C)$. Here ρ_0 is the density of the mixture for the average values of the temperature and concentration, T and C are small deviations from the average values, β_1 is the thermal-expansion coefficient of the mixture, and β_2 is the density concentration coefficient ($\beta_2 > 0$ since C is the concentration of the light component). The motion of the mixture is described by the system [1]

$$\begin{gathered}
\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -(1/\rho_0)\nabla p + \nu\Delta\boldsymbol{u} - \boldsymbol{g}(\beta_1 T + \beta_2 C),\\
T_t + \boldsymbol{u} \cdot \nabla T = \chi\Delta T,\\
C_t + \boldsymbol{u} \cdot \nabla C = d\,\Delta C + \alpha d\,\Delta T,\\
\operatorname{div}\boldsymbol{u} = 0,
\end{gathered} \tag{1}$$

where $\boldsymbol{u}$ is the velocity, p is the pressure deviation from the hydrostatic value, ν are χ are the kinematic viscosity and thermal diffusivity of the mixture, respectively, d is the diffusion coefficient, α is the thermal-diffusion parameter, and $\boldsymbol{g}$ is the free-fall acceleration. It is assumed that all characteristics of the medium are constant and correspond

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to the average values of the temperature and concentration. In the case of normal thermal diffusion, we have $\alpha < 0$, and for anomalous thermal diffusion, $\alpha > 0$.

In the literature there are a number of papers dealing with constructing exact solutions of system (1) and studying their stability. In particular, the stability of convective flows of a binary mixture in a vertical channel in the presence of longitudinal concentration and (or) temperature gradients ignoring thermal diffusion was studied in [2, 3]. The effect of thermal diffusion on the stability of a vertical layer with a constant temperature difference between the walls was investigated in [4, 5] (in the layer there was also a longitudinal concentration gradient). The instability of a plane horizontal layer of an incompressible binary gas mixture subjected to a time-dependent transverse temperature gradient was studied in [6].

The group properties of Eqs. (1) in the case $\boldsymbol{g} = 0$ were studied in [7], where an exact invariant solution was constructed that describes the motion of two mixtures with a common interface. However, a systematic study of the examined system using group analysis methods has not been performed. An exception is a paper [8], in which a group analysis of the three-dimensional equations (1) was performed.

In the present paper, we consider the case of plane motion, for which it is necessary to set $\boldsymbol{x} = (x^1, x^2)$, $\boldsymbol{u} = (u^1, u^2)$, and $\boldsymbol{g} = (0, -g)$, where g is the acceleration of gravity. The group properties of the corresponding equations (1) are studied, and the invariant solutions are classified (optimal systems of subalgebras are constructed). Examples of the exact invariant solutions are given, and their physical interpretation is analyzed. It is shown that the solutions describe thermal diffusion in an inclined layer with a free boundary and in a vertical layer with solid walls.

1. Group Properties of the Equations of the Model. We consider the problem of finding a transformation group that does not change system (1). In Lie theory, each transformation group is put in correspondence to a Lie algebra of differential operators. Calculations show that the two-dimensional equations (1) admit the Lie algebra L that is represented as the semidirect sum $L = L_4 \oplus L_\infty$. The finite-dimensional subalgebra L_4 is formed by the operators

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{1}{\beta_1}\frac{\partial}{\partial T} - \frac{1}{\beta_2}\frac{\partial}{\partial C}, \qquad X_3 = \rho_0 g x^2 \frac{\partial}{\partial p} + \frac{1}{\beta_2}\frac{\partial}{\partial C}, \\ X_4 = 2t\frac{\partial}{\partial t} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2} - u^1\frac{\partial}{\partial u^1} - u^2\frac{\partial}{\partial u^2} - 2p\frac{\partial}{\partial p} - 3T\frac{\partial}{\partial T} - 3C\frac{\partial}{\partial C}, \end{gathered} \tag{2}$$

and the infinite-dimensional ideal L_∞ has the basis

$$H_1(f^1(t)) = f^1(t)\frac{\partial}{\partial x^1} + f^1_t(t)\frac{\partial}{\partial u^1} - \rho_0 x^1 f^1_{tt}(t)\frac{\partial}{\partial p},$$

$$H_2(f^2(t)) = f^2(t)\frac{\partial}{\partial x^2} + f^2_t(t)\frac{\partial}{\partial u^2} - \rho_0 x^2 f^2_{tt}(t)\frac{\partial}{\partial p},$$

$$H_0(f^0(t)) = f^0(t)\frac{\partial}{\partial p},$$

where $f^i(t)$ $(i = 0, 1, 2)$ are arbitrary smooth functions. If the constants included in the system are linked by the relation $\alpha = \beta_1(d - \chi)/(\beta_2 d)$ $(d \neq \chi)$, the equations also admit the operator

$$X_5 = T\frac{\partial}{\partial T} - \frac{\beta_1}{\beta_2}T\frac{\partial}{\partial C}.$$

Below, it is assumed that this relation does not hold and that the operator X_5 is not admitted. System (1) also has discrete symmetries

$$\begin{gathered} d_1\colon \qquad \tilde{x}^1 = -x^1, \quad \tilde{u}^1 = -u^1, \\ d_2\colon \qquad \tilde{x}^2 = -x^2, \quad \tilde{u}^2 = -u^2, \quad \tilde{T} = -T, \quad \tilde{C} = -C. \end{gathered} \tag{3}$$

2. Optimal Systems of Subalgebras. It is known that to seek substantially different invariant solutions (with respect to the action of the admissible transformation group), it is necessary to construct optimal systems of subalgebras for the corresponding Lie operator algebra [9–11]. We construct such systems for the Lie algebra L. The optimal system of subalgebras of order k for the algebra L is denoted by $\Theta_k L$.

TABLE 1

Optimal System of Subalgebras ΘL_4

i	Basis P_i	Nor P_i	i	Basis P_i	Nor P_i
1	1, 2, 3, 4	= 1	11	1, 4	= 11
2	1, 2, 3	1	12	2, 4	= 12
3	1, 2, 4	= 3	13	$\lambda 2 + 3$, 4	= 13
4	2, 3, 4	= 4	14	1	1
5	1, $\lambda 2 + 3$, 4	= 5	15	2	1
6	1, 2	1	16	$\lambda 2 + 3$	1
7	2, 3	1	17	$1 + 2$	2
8	1, $\lambda 2 + 3$	1	18	$1 + \lambda 2 + 3$	2
9	$1 + 2$, $\lambda 1 + 3$	2	19	4	= 19
10	$1 + 3$, 2	2	20	0	1

Note. The "=" sign denotes self-normalized subalgebras.

As a first step, we construct the optimal system for the finite-dimensional algebra L_4 using its decomposition into the semidirect sum $L_4 = J \oplus N$ of the eigenideal $J = \{X_1, X_2\}$ and the subalgebra $N = \{X_3, X_4\}$. The optimal system ΘL_4 is given in Table 1. The first column gives the subalgebra numbers. The second column gives the subalgebra bases, which are written symbolically using the corresponding operator numbers. The notation $\lambda 2 + 3$ means $\lambda X_2 + X_3$, etc; the constant λ takes any real values. The third column gives the subalgebra normalizer number in L_4 (the equality sign indicates the self-normalized subalgebras). In constructing the optimal system, we take into account the automorphisms generated by the discrete symmetries (3).

As a second step, we construct the optimal first- and second-order systems for the algebra L which is infinite-dimensional. The operator of the general form belonging to the ideal L_∞ is written as

$$H(\boldsymbol{f}) = H_1(f^1) + H_2(f^2) + H_0(f^0), \qquad \boldsymbol{f}(t) = (f^1(t), f^2(t), f^0(t)).$$

To construct $\Theta_1 L$, it is necessary to classify the subalgebras from the following two classes:

$$1) \quad \{H(\boldsymbol{f})\};$$
$$2) \quad \{P + H(\boldsymbol{f})\}, \qquad \{P\} \in \Theta_1 L_4.$$

The first class lies in the ideal L_∞, and the second has zero intersection with this ideal. The subalgebras with the basis operator P are taken from the optimal first-order system $\Theta_1 L_4$ (see Table 1). The finding of $\Theta_2 L$ reduces to classifying the subalgebras from the following three classes:

$$1) \quad \{H(\boldsymbol{f}),\, H(\boldsymbol{g})\};$$
$$2) \quad \{P + H(\boldsymbol{f}),\, H(\boldsymbol{g})\}, \qquad \{P\} \in \Theta_1 L_4;$$
$$3) \quad \{P + H(\boldsymbol{f}),\, Q + H(\boldsymbol{g})\}, \qquad \{P,\, Q\} \in \Theta_2 L_4.$$

Here the first class belongs to the ideal L_∞, and the second and third classes have one-dimensional and zero intersections with the ideal L_∞, respectively. The subalgebras $\{P, Q\}$ are taken from the optimal second-order system $\Theta_2 L_4$.

The optimal system $\Theta_1 L$ is given in Table 2. The invariant solutions are constructed using the finite-dimensional subalgebras from the optimal second-order system $\Theta_2 L$, which are given in Table 3. The bases of the subalgebras are indicated in the second column. The constants λ, μ, γ, and δ take any real values unless otherwise specified.

3. Construction of the Exact Solutions. We consider examples of the invariant solutions constructed for the subalgebras from the optimal system $\Theta_2 L$ (see Table 3). The use of two-dimensional subalgebras reduces integration of the constitutive system to solution of ordinary differential equations. Below, we use the standard notation of the coordinate vectors $\boldsymbol{x} = (x, y)$ and velocity $\boldsymbol{u} = (u, v)$.

TABLE 2

Optimal System of Subalgebras $\Theta_1 L$

i	Basis	Note
1	X_1	—
2	$X_1 + X_2$	
3	$X_1 + \lambda X_2 + X_3$	
4	X_4	
5	$H_0(f^0)$	$f^0 \not\equiv 0$
6	$X_2 + H_0(f^0)$	$f^0 \not\equiv 0$
7	$H_1(f^1) + H_2(f^2)$	—
8	$X_2 + H_1(f^1) + H_2(f^2)$	
9	$\lambda X_2 + X_3 + H_1(f^1) + H_2(f^2)$	

Example No. 1. We consider subalgebra 4 with the basis

$$X_1 = \frac{\partial}{\partial t}, \qquad H_1(1) + H_2(\lambda) = \frac{\partial}{\partial x} + \lambda \frac{\partial}{\partial y}.$$

The corresponding invariant solution has the form

$$u = u(\xi), \quad v = v(\xi), \quad p = p(\xi), \quad T = T(\xi), \quad C = C(\xi), \quad \xi = y - \lambda x.$$

Substituting this solution into system (1), from the continuity equation we have $v = \lambda u + v_0$, where $v_0 = \text{const}$. If $v_0 = 0$, the required functions satisfy the system

$$\nu(\lambda^2+1)^2 u_{\xi\xi} + \lambda g(\beta_1 T + \beta_2 C) = 0, \quad p_\xi + (\rho_0 \nu/\lambda)(\lambda^2+1) u_{\xi\xi} = 0, \quad T_{\xi\xi} = C_{\xi\xi} = 0,$$

which is easily integrated:

$$u = -\frac{\lambda g}{6\nu(\lambda^2+1)^2}\Big((\beta_1 c_1 + \beta_2 c_3)\xi^3 + 3(\beta_1 c_2 + \beta_2 c_4)\xi^2\Big) + c_5\xi + c_6, \qquad v = \lambda u,$$
$$p = \frac{\rho_0 g}{2(\lambda^2+1)}\Big((\beta_1 c_1 + \beta_2 c_3)\xi^2 + 2(\beta_1 c_2 + \beta_2 c_4)\xi\Big) + c_7, \tag{4}$$
$$T = c_1\xi + c_2, \qquad C = c_3\xi + c_4, \qquad \xi = y - \lambda x.$$

Here $c_1, \ldots, c_7$ are arbitrary constants. Below, we use the same notation for arbitrary constants.

Example No. 2. We consider subalgebra 29 with the basis

$$X_1 = \frac{\partial}{\partial t}, \qquad \lambda X_2 + X_3 \pm H_2(1) = \frac{\lambda}{\beta_1}\frac{\partial}{\partial T} + \frac{1-\lambda}{\beta_2}\frac{\partial}{\partial C} + \rho_0 g y \frac{\partial}{\partial p} \pm \frac{\partial}{\partial y}.$$

The invariant solution in this case is written as

$$u = U(x), \quad v = V(x), \quad p = P(x) \pm \frac{\rho_0 g}{2} y^2, \quad T = \tilde{T}(x) \pm \frac{\lambda}{\beta_1} y, \quad C = \tilde{C}(x) \pm \frac{1-\lambda}{\beta_2} y.$$

Substituting the solution into the constitutive system, from the continuity equation we have $U = U_0 = \text{const}$. Then, from the first equation of system (1) it follows that $p = p_0 = \text{const}$. Subsequently, the case $U_0 = 0$, $\lambda \neq 0$ is considered. The new unknown functions satisfy the system

$$\nu V_{xx} + g(\beta_1 \tilde{T} + \beta_2 \tilde{C}) = 0; \tag{5}$$

$$\chi \tilde{T}_{xx} \mp \lambda V/\beta_1 = 0; \tag{6}$$

$$d\tilde{C}_{xx} + \alpha d\tilde{T}_{xx} \mp (1-\lambda)V/\beta_2 = 0. \tag{7}$$

We express the function V from (6) and substitute it into Eqs. (5) and (7). Then, double integration of d Eq. (7) yields

$$\tilde{T}_{xxxx} \pm \lambda g(\beta_1 \tilde{T} + \beta_2 \tilde{C})/(\beta_1 \nu \chi) = 0; \tag{8}$$

$$\tilde{C} = \Big(\beta_1 \chi(1-\lambda)/(\beta_2 d\lambda) - \alpha)\tilde{T} + \tilde{c_1} x + \tilde{c_2}. \tag{9}$$

TABLE 3

Finite-Dimensional Subalgebras from the Optimal System $\Theta_2 L$

i	Basis	Note
1	$X_1,\ H_0(1)$	—
2	$X_1,\ H_2(1)$	—
3	$X_1,\ H_1(1)+H_0(1)$	—
4	$X_1,\ H_1(1)+H_2(\lambda)$	$\lambda \geq 0$
5	$X_1,\ H_0(\mathrm{e}^{\pm t})$	—
6	$X_1,\ H_2(\mathrm{e}^{\pm t})$	—
7	$X_1,\ H_1(\mathrm{e}^{\pm t})+H_2(\lambda\,\mathrm{e}^{\pm t})$	$\lambda \geq 0$
8	$X_1+X_2,\ H_0(1)$	—
9	$X_1+X_2,\ H_1(1)+H_0(\lambda)$	$\lambda \geq 0$
10	$X_1+X_2,\ H_1(\lambda)+H_2(1)$	$\lambda \geq 0$
11	$X_1+\lambda X_2,\ H_0(\mathrm{e}^{\pm t})$	$\lambda > 0$
12	$X_1+\lambda X_2,\ H_2(\mathrm{e}^{\pm t})$	$\lambda > 0$
13	$X_1+\lambda X_2,\ H_1(\mathrm{e}^{\pm t})+H_2(\mu\,\mathrm{e}^{\pm t})$	$\lambda > 0,\ \mu \geq 0$
14	$X_1+\lambda X_2+X_3,\ H_1(1)+H_0(\mu)$	$\mu \geq 0$
15	$X_1+\lambda X_2+X_3,\ H_1(\mu)+H_2(1)+H_0(\rho_0 g t)$	$\mu \geq 0$
16	$X_1+\lambda X_2+\mu X_3,\ H_0(\mathrm{e}^{\pm t})$	$\mu > 0$
17	$X_1+\lambda X_2+\mu X_3,\ H_1(\mathrm{e}^{\pm t})$	$\mu > 0$
18	$X_1+\lambda X_2+\mu X_3,\ H_1(\delta\,\mathrm{e}^{\pm t})+H_2(\mathrm{e}^{\pm t})+H_0(\mu\rho_0 g t\,\mathrm{e}^{\pm t})$	$\mu > 0,\ \delta \geq 0$
19	$X_4,\ H_0(t^\gamma)$	—
20	$X_4,\ H_2(t^\gamma)$	$\gamma \neq 1/2$
21	$X_4,\ H_1(t^\gamma)+H_2(\lambda t^\gamma)$	$\gamma \neq 1/2,\ \lambda \geq 0$
22	$X_4,\ H_2(\sqrt{t})+H_0(\lambda/t)$	$\lambda \geq 0$
23	$X_4,\ H_1(\sqrt{t})+H_2(\mu\sqrt{t})+H_0(\lambda/t)$	$\lambda \geq 0,\ \mu \geq 0$
24	$X_1,\ X_2$	—
25	$X_1,\ X_2+H_0(1)$	—
26	$X_1,\ X_2+H_1(1)+H_0(\lambda)$	$\lambda \geq 0$
27	$X_1,\ X_2+H_1(\lambda)\pm H_2(1)$	$\lambda \geq 0$
28	$X_1,\ \lambda X_2+X_3$	—
29	$X_1,\ \lambda X_2+X_3\pm H_2(1)$	—
30	$X_1,\ \lambda X_2+X_3+H_1(1)+H_2(\mu)$	—
31	$X_1+X_2,\ \lambda X_1+X_3+H_1(\mu)+H_2(\delta)$	$\mu \geq 0$
32	$X_1+X_3,\ X_2+H_1(\lambda)+H_0(\mu)$	$\lambda \geq 0$
33	$X_1+X_3,\ X_2+H_1(\lambda)+H_2(\mu)+H_0(\mu\rho_0 g t)$	$\lambda \geq 0,\ \mu \neq 0$
34	$X_1,\ X_4$	—
35	$X_2+H_0(\lambda\sqrt{t}),\ X_4$	$\lambda > 0$
36	$X_2+H_1(\lambda t^2)+H_2(\mu t^2),\ X_4$	$\lambda \geq 0$
37	$\lambda X_2+X_3+H_2(4gt^2/9)+H_0(\mu\sqrt{t}),\ X_4$	$\mu > 0$
38	$\lambda X_2+X_3+H_1(\mu t^2)+H_2(\delta t^2),\ X_4$	$\mu \geq 0$

Substitution of the function C from (9) into (8) gives

$$\tilde{T}_{xxxx} \pm \frac{g(\beta_1(\lambda d + (1-\lambda)\chi) - \beta_2\lambda\alpha d)}{\beta_1\nu\chi d}\,\tilde{T} \pm \frac{\beta_2 g\lambda}{\beta_1\nu\chi}\,(\tilde{c_1}x + \tilde{c_2}) = 0. \tag{10}$$

The coefficients at $\tilde{T}$ in (10) are denoted by

$$a = \pm g(\beta_1(\lambda d + (1-\lambda)\chi) - \beta_2\lambda\alpha d)/(\beta_1\nu\chi d).$$

Equation (10) has three different solutions for the cases $a < 0$, $a > 0$, and $a = 0$ [8].The functions V and $\tilde{C}$ are determined from Eqs. (6) and (9), respectively.

Case 1: $a < 0$. Setting $\gamma = \pm\sqrt[4]{-a}$, we write the solution of the constitutive system as follows:

$$u = 0, \quad v = \pm(\beta_1\chi\gamma^2/\lambda)(c_3\cosh\,\gamma x + c_4\sinh\,\gamma x - c_5\cos\gamma x - c_6\sin\gamma x), \quad p = p_0 \pm \rho_0 g y^2/2,$$

$$T = c_3\cosh\,\gamma x + c_4\sinh\,\gamma x + c_5\cos\gamma x + c_6\sin\gamma x + \beta_2(c_1x + c_2) \pm (\lambda/\beta_1)y, \tag{11}$$

$$C = \Big(\frac{\beta_1\chi(1-\lambda)}{\beta_2\lambda d} - \alpha\Big)(c_3\cosh\,\gamma x + c_4\sinh\,\gamma x + c_5\cos\gamma x + c_6\sin\gamma x) - \beta_1(c_1x + c_2) \pm \frac{1-\lambda}{\beta_2}\,y.$$

The new constants c_1 and c_2 are defined by the formulas

$$c_i = -\frac{\lambda d}{\beta_1(\lambda d + (1-\lambda)\chi) - \beta_2\lambda\alpha d}\,\tilde{c_i}, \quad i = 1, 2. \tag{12}$$

Case 2: $a > 0$. Setting $\gamma = \pm\sqrt[4]{a/4}$, we obtain the following representation of the solution of the constitutive system:

$$\begin{gathered} u = 0, \qquad p = p_0 \pm \rho_0 g y^2/2, \\ v = \pm(2\beta_1\chi\gamma^2/\lambda)(c_4\sinh\,\gamma x\cos\gamma x - c_3\sinh\,\gamma x\sin\gamma x + c_6\cosh\,\gamma x\cos\gamma x - c_5\cosh\,\gamma x\sin\gamma x), \\ T = c_3\cosh\,\gamma x\cos\gamma x + c_4\cosh\,\gamma x\sin\gamma x + c_5\sinh\,\gamma x\cos\gamma x + c_6\sinh\,\gamma x\sin\gamma x + \beta_2(c_1x + c_2) \pm \lambda y/\beta_1, \\ C = (\beta_1\chi(1-\lambda)/(\beta_2 d\lambda) - \alpha)(c_3\cosh\,\gamma x\cos\gamma x + c_4\cosh\,\gamma x\sin\gamma x + c_5\sinh\,\gamma x\cos\gamma x + c_6\sinh\,\gamma x\sin\gamma x) \\ -\beta_1(c_1x + c_2) \pm (1-\lambda)y/\beta_2. \end{gathered} \tag{13}$$

In this case, the constants are also changed by formula (12).

Case 3: $a = 0$. In this case, the solution has the form

$$\begin{gathered} u = 0, \qquad v = (\beta_1(\chi - d) + \beta_2\alpha d)\Big(\frac{c_5}{6}\,x^3 + \frac{c_4}{2}\,x^2 \pm c_3x \pm c_2\Big), \qquad p = p_0 \pm \frac{\rho_0 g}{2}\,y^2, \\ T = \pm\frac{c_5}{120}\,x^5 \pm \frac{c_4}{24}\,x^4 + \frac{c_3}{6}\,x^3 + \frac{c_2}{2}\,x^2 + c_1x + c_0 \pm \frac{\chi}{\beta_1(\chi - d) + \beta_2\alpha d}\,y, \\ C = -\frac{\beta_1}{\beta_2}\Big(\pm\frac{c_5}{120}\,x^5 \pm \frac{c_4}{24}\,x^4 + \frac{c_3}{6}\,x^3 + \frac{c_2}{2}\,x^2 + c_1x + c_0\Big) - \\ -\frac{\nu(\beta_1(\chi - d) + \beta_2\alpha d)}{g\beta_2}\,(c_5x + c_4) \pm \frac{d(\alpha\beta_2 - \beta_1)}{\beta_2(\beta_1(\chi - d) + \beta_2\alpha d)}\,y. \end{gathered} \tag{14}$$

Here the new constants c_5 and c_4 are defined by the formulas

$$c_5 = -\frac{g\beta_2}{\nu(\beta_1(\chi - d) + \beta_2\alpha d)}\,\tilde{c_1}, \qquad c_4 = -\frac{g\beta_2}{\nu(\beta_1(\chi - d) + \beta_2\alpha d)}\,\tilde{c_2}.$$

Example No. 3. We consider subalgebra 6, whose basis operators are written as

$$X_1 = \frac{\partial}{\partial t}, \qquad H_2(\mathrm{e}^{\pm t}) = \mathrm{e}^{\pm t}\,\frac{\partial}{\partial y} \pm \mathrm{e}^{\pm t}\,\frac{\partial}{\partial v} - \rho_0\,\mathrm{e}^{\pm t}\,y\,\frac{\partial}{\partial p}.$$

Here the invariant solution is represented as

$$u = U(x), \quad v = V(x) \pm y, \quad p = P(x) - \rho_0 y^2/2, \quad T = T(x), \quad C = C(x).$$

Substituting the given solution into the constitutive system, from the continuity equation we find the velocity $U = \mp(x + c_1)$, where c_1 is an arbitrary constant. Then, from the first equation of system (1), we obtain the function P in the pressure expression:

$$P(x) = -\rho_0 x^2/2 - c_1\rho_0 x + c_2.$$

The remaining functions satisfy the system

$$\nu V_{xx} \pm (x + c_1)V_x \mp V + g(\beta_1 T + \beta_2 C) = 0; \tag{15}$$

$$\chi T_{xx} \pm (x + c_1)T_x = 0; \tag{16}$$

$$dC_{xx} \pm (x + c_1)C_x + \alpha d T_{xx} = 0. \tag{17}$$

We introduce the function

$$F^{\mp}(x, a) = \int\limits_0^{(x+c_1)/\sqrt{2a}} \exp(\mp s^2)\, ds.$$

System (16), (17) is integrated (see [12]):

$$T = c_3 + c_4 F^{\mp}(x, \chi), \qquad C = c_5 + c_6 F^{\mp}(x, d) + \alpha d T/(\chi - d). \tag{18}$$

Next, we seek a solution of the homogeneous equation (15), which according to [12], is written

$$V = c_7[\exp(\mp(x + c_1)^2/(2\nu)) \pm \sqrt{2/\nu}\,(x + c_1)F^{\mp}(x, \nu)] + c_8(x + c_1)/\sqrt{2\nu}.$$

The solution of the corresponding inhomogeneous equation is found by substitution of expressions (18) into (15). As a result, we obtain an example of the exact solution of system (1) in the form

$$u = \mp(x + c_1),$$

$$v = c_7[\exp(\mp(x + c_1)^2/(2\nu)) \pm \sqrt{2/\nu}\,(x + c_1)F^{\mp}(x, \nu)] + c_8(x + c_1)/\sqrt{2\nu}$$

$$+ g(\beta_1(\chi - d) + \beta_2\alpha d)(\pm c_3 - c_4 G(x, \chi))/(\chi - d) + \beta_2 g(\pm c_5 - c_6 G(x, d)) \pm y, \tag{19}$$

$$p = -\rho_0(x^2 + y^2)/2 - c_1\rho_0 x + c_2, \quad T = c_3 + c_4 F^{\mp}(x, \chi), \quad C = c_5 + c_6 F^{\mp}(x, d) + \alpha d T/(\chi - d).$$

Here

$$G(x, a) = \Big[\frac{x + c_1}{\sqrt{\nu a}}\, F^{\mp}(x, \nu) \pm \frac{1}{\sqrt{2a}}\, \exp\Big(\mp\frac{1}{2\nu}\,(x + c_1)^2\Big)\Big] \int\limits_{-c_1}^{x} \exp\Big(\pm\frac{a - \nu}{2\nu a}\,(\tau + c_1)^2\Big)\, d\tau$$

$$- \frac{x + c_1}{\sqrt{\nu a}} \int\limits_{-c_1}^{x} \exp\Big(\pm\frac{a - \nu}{2\nu a}\,(\tau + c_1)^2\Big) F^{\mp}(\tau, \nu)\, d\tau \mp F^{\mp}(x, a).$$

4. Physical Interpretation of the Solutions. We give a possible physical interpretation of the solutions obtained in example Nos. 1 and 2 from Sec. 3. The interpretation of solution (19), found in example No. 3 is difficult.

Thermal Diffusion in an Inclined Layer. We consider solution (4), in which all unknown functions remain constant on the straight lines $\xi = y - \lambda x = \text{const}$. Let a liquid layer of thickness h be located at an angle $0 \le \varphi < 90°$ to the horizon. From below and from above, the liquid is bounded by a heated solid wall and a free boundary, respectively; they are straight lines with a normal unit vector $\boldsymbol{n}$ (Fig. 1). It is assumed that in any cross section of the layer there is a constant temperature difference Θ between the solid wall and the free boundary. On the wall $y - x\tan\varphi = 0$, we impose the conditions of attachment and absence of material flow and specify the temperature distribution:

$$u = v = 0, \qquad -d\Big(\frac{\partial C}{\partial \boldsymbol{n}} + \alpha\,\frac{\partial T}{\partial \boldsymbol{n}}\Big) = 0, \qquad T = \Theta.$$

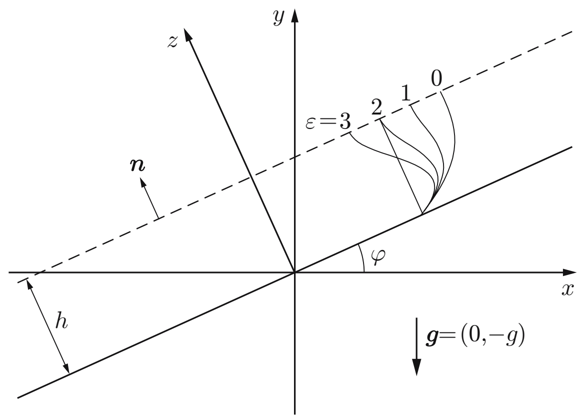

Fig. 1. An inclined layer of a liquid.

On the free boundary, $y - x\tan\varphi = h/\cos\varphi$, the following kinematic and dynamic conditions should be satisfied:

$$u\tan\varphi - v = 0, \qquad ((p - p_g)E - 2\nu\rho_0 D(\boldsymbol{u}))\boldsymbol{n} = 2\sigma H\boldsymbol{n} + \nabla_\Gamma\sigma. \tag{20}$$

Here p_g is the pressure on the free boundary, E is the unit matrix, $D(\boldsymbol{u})$ is the strain-rate tensor, $\sigma = \sigma(T, C)$ is the Surface-tension coefficient, H is the mean curvature of the free surface, and $\nabla_\Gamma = \nabla - \boldsymbol{n}(\boldsymbol{n}\cdot\nabla)$ is the surface gradient. Since in solution (4), the temperature and concentration on the free boundary are constant, the surface gradient in (20) is equal to zero. The free surface is a straight line; therefore $H = 0$. Here the kinematic condition is satisfied identically. In addition, on the free boundary, the temperature distribution is specified and the condition of no material flow through the boundary is imposed:

$$T = 0, \qquad -d\Big(\frac{\partial C}{\partial \boldsymbol{n}} + \alpha\,\frac{\partial T}{\partial \boldsymbol{n}}\Big) = 0.$$

We specify the average concentration in the cross section and assume that it remains constant along the layer. Then, the function C, which defines the deviations from the average values, should obey the condition

$$\int\limits_0^h C\,d\gamma = 0, \qquad \gamma\colon\quad y\tan\varphi + x = 0.$$

To determine the unknown constants, we write solution (4) and the boundary conditions in nondimensional form. Introducing the characteristic scales of time h^2/ν, distance h, velocity $g\beta_1\Theta h^2/\nu$, pressure $\rho_0 gh\beta_1\Theta$, temperature Θ, and concentration $\beta_1\Theta/\beta_2$, we write Eq. (1) in dimensionless variables:

$$\boldsymbol{u}_t + \mathrm{Gr}\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + \Delta\boldsymbol{u} + \boldsymbol{q}(T + C),$$

$$T_t + \mathrm{Gr}\,(\boldsymbol{u}\cdot\nabla T) = \Delta T/\mathrm{Pr}\,,$$

$$C_t + \mathrm{Gr}\,(\boldsymbol{u}\cdot\nabla C) = (\Delta C - \varepsilon\Delta T)/\mathrm{Sc},$$

$$\mathrm{div}\,\boldsymbol{u} = 0,$$

where $\boldsymbol{q} = (0, 1)$. The system contains four dimensionless parameters — the Grashof number $\mathrm{Gr} = g\beta_1\Theta h^3/\nu^2$, the Prandtl number $\mathrm{Pr} = \nu/\chi$, the Schmidt number $\mathrm{Sc} = \nu/d$, and the parameter $\varepsilon = -\alpha\beta_2/\beta_1$, which determines the thermal diffusion effect.

Since the solution considered depends on one variable, it is convenient to introduce the z axis perpendicular to the layer and to write the quantities as functions of the z coordinate. Solution (4) which satisfies the boundary conditions is written in the dimensionless variables as follows:

$$u' = \sin\varphi(2(\varepsilon + 1)z^3 - 3(\varepsilon + 2)z^2 + 6z)/12,$$

$$p = \cos\varphi(-(\varepsilon + 1)z^2 + (\varepsilon + 2)z - 1)/2 + p_g', \quad T = -z + 1, \quad C = \varepsilon(-z + 1/2).$$

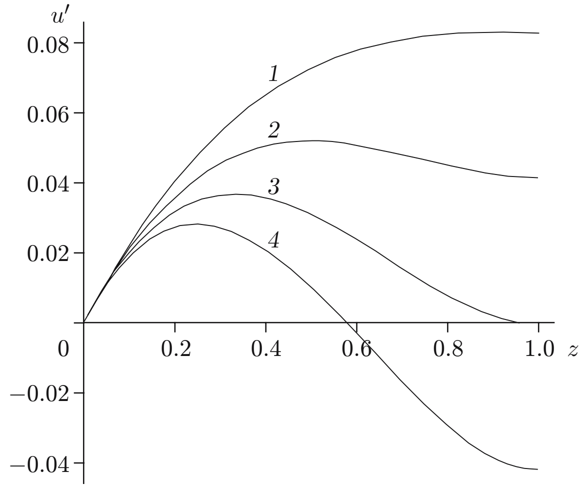

Fig. 2

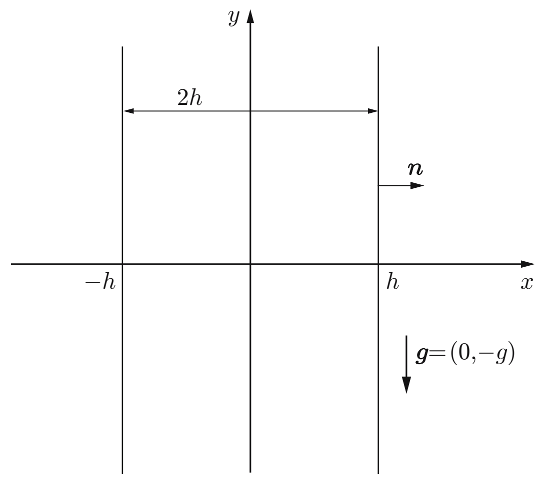

Fig. 3

Fig. 2. Velocity profiles in an inclined layer for various values of the thermal-diffusion parameter: $\varepsilon = 0$ (1), 1 (2), 2 (3), and 3 (4).

Fig. 3. Vertical layer.

Here the function $u' = u/\cos\varphi$ defines the velocity profile in the cross section, $p'_g = p_g(\rho_0 g h \beta_1 \Theta)^{-1}$. Figure 2 gives velocity profiles for various values of the thermal-diffusion parameter at an inclination angle of $\varphi = 30°$. The straight line $z = 0$ corresponds to the solid wall, and $z = 1$ to the free boundary. In the absence of thermal diffusion (curve 1), the liquid rises up along the layer as a result of the temperature difference between the wall and the free boundary. In the case of anomalous thermal diffusion, the parameter $\varepsilon < 0$ and the light component are concentrated at the cold free boundary. As a result, the rate of rise of the liquid increases but its profile is similar to the velocity profile for $\varepsilon = 0$. For $\varepsilon > 0$, normal thermal diffusion takes place. The light component diffuses toward the heated wall, and the heavier component toward the cold boundary. This results in a decrease in the velocity (curve 2). For $\varepsilon = 2$ (curve 3), the velocity on the free boundary vanishes. With a further increase in the thermal-diffusion parameter, the concentration of the heavier component at the free boundary grows. Under the action of gravity, the liquid starts moving downward, but at the wall, the opposite direction of motion (curve 4) is preserved.

Thermal Diffusion in a Vertical Layer. We consider solution (11), (13), (14) and give its possible physical interpretation. Let a vertical fluid layer of thickness $2h$ be enclosed between two solid walls with a unit normal vector $\boldsymbol{n}$ (Fig. 3).The conditions of attachment and no material flow through the wall are imposed on the walls, and a linear temperature distribution is specified on the y coordinate. It is assumed that in any cross section there is a constant temperature difference 2Θ. Thus, the conditions on the walls $x = \pm h$ has the form

$$u = v = 0, \qquad T = Ay \pm \Theta, \qquad -d\Big(\frac{\partial C}{\partial \boldsymbol{n}} + \alpha\,\frac{\partial T}{\partial \boldsymbol{n}}\Big) = 0. \tag{21}$$

In addition, it is postulated that the flow is closed and the vertical concentration gradient is constant:

$$\int\limits_{-h}^{h} v\,dx = 0, \qquad \lim_{l\to\infty} \frac{1}{2l} \int\limits_{-l}^{l} \frac{\partial C}{\partial y}\,dy = B. \tag{22}$$

Conditions (21) and (22) are written in dimensionless form

$$x = \pm 1: \qquad u = v = 0, \qquad T = \frac{\mathrm{Ra}}{\mathrm{GrPr}}\,y \pm 1, \qquad \frac{\partial C}{\partial x} - \varepsilon\,\frac{\partial T}{\partial x} = 0,$$

$$\int\limits_{-1}^{1} v\,dx = 0, \qquad \lim_{l\to\infty} \frac{1}{2l} \int\limits_{-l}^{l} \frac{\partial C}{\partial y}\,dy = \frac{\mathrm{Ra}_d}{\mathrm{GrSc}}.$$

Here we use the same dimensionless variables as in the previous example but introduce two new dimensionless parameters — the Rayleigh number $\mathrm{Ra} = g\beta_1 A h^4/(\nu\chi)$ and the Rayleigh concentration number $\mathrm{Ra}_d = g\beta_2 B h^4/(\nu d)$. These numbers are determined from the vertical temperature and concentration gradients, respectively.

To satisfy the specified boundary conditions, the solution considered is subjected to an expansion transformation specified by the operator X_4 from (2). This allows us to introduce an independent real parameter into the solutions that defines the coefficient at y in the concentration expressions [see (11) and (13)].

In the dimensionless variables, the parameter a, which defines the type of solution, becomes $a' = \mathrm{Ra}\,(\varepsilon + 1) + \mathrm{Ra}_d$. We write a solution that satisfies the specified boundary conditions for all three cases.

Case 1: $\mathrm{Ra}\,(\varepsilon+1) + \mathrm{Ra}_d < 0$. The solution of system (1) is given by the formulas

$$\begin{gathered}
u = 0, \qquad v = \frac{(\varepsilon+1)\gamma^2}{S}\Big[\frac{\sinh\gamma x}{\sinh\gamma} - \frac{\sin\gamma x}{\sin\gamma}\Big], \qquad p = p_0 + \frac{1}{2}\Big[\frac{\mathrm{Ra}}{\mathrm{GrPr}} + \frac{\mathrm{Ra}_d}{\mathrm{GrSc}}\Big]y^2, \\
T = \frac{(\varepsilon+1)\,\mathrm{Ra}}{S}\Big[\frac{\sinh\gamma x}{\sinh\gamma} + \frac{\sin\gamma x}{\sin\gamma}\Big] + \frac{\gamma\,\mathrm{Ra}_d}{S}\,(\cot\gamma + \coth\gamma)x + \frac{\mathrm{Ra}}{\mathrm{GrPr}}\,y, \\
C = \frac{(\varepsilon+1)(\mathrm{Ra}\,\varepsilon + \mathrm{Ra}_d)}{S}\Big[\frac{\sinh\gamma x}{\sinh\gamma} + \frac{\sin\gamma x}{\sin\gamma}\Big] - \frac{\gamma\,\mathrm{Ra}_d}{S}\,(\cot\gamma + \coth\gamma)x + \frac{\mathrm{Ra}_d}{\mathrm{GrSc}}\,y, \\
\gamma = \sqrt[4]{-\mathrm{Ra}\,(\varepsilon+1) - \mathrm{Ra}_d}, \qquad S = 2\,\mathrm{Ra}\,(\varepsilon+1) + \gamma\,\mathrm{Ra}_d\,(\cot\gamma + \coth\gamma).
\end{gathered} \tag{23}$$

Case 2: $\mathrm{Ra}\,(\varepsilon+1) + \mathrm{Ra}_d > 0$. The required functions are written as

$$\begin{gathered}
u = 0, \qquad v = \frac{4(\varepsilon+1)\gamma^2}{S}\,(\sin\gamma\cosh\gamma\cos\gamma x\sinh\gamma x - \cos\gamma\sinh\gamma\sin\gamma x\cosh\gamma x), \\
T = \frac{2(\varepsilon+1)\,\mathrm{Ra}}{S}\,(\cos\gamma\sinh\gamma\cos\gamma x\sinh\gamma x + \sin\gamma\cosh\gamma\sin\gamma x\cosh\gamma x) \\
+ \frac{\gamma\,\mathrm{Ra}_d}{S}\,(\sin 2\gamma + \sinh 2\gamma)x + \frac{\mathrm{Ra}}{\mathrm{GrPr}}\,y, \\
p = p_0 + \frac{1}{2}\Big[\frac{\mathrm{Ra}}{\mathrm{GrPr}} + \frac{\mathrm{Ra}_d}{\mathrm{GrSc}}\Big]y^2, \\
C = \frac{2(\varepsilon+1)(\mathrm{Ra}\,\varepsilon + \mathrm{Ra}_d)}{S}\,(\cos\gamma\sinh\gamma\cos\gamma x\sinh\gamma x + \sin\gamma\cosh\gamma\sin\gamma x\cosh\gamma x) \\
- \frac{\gamma\,\mathrm{Ra}_d}{S}\,(\sin 2\gamma + \sinh 2\gamma)x + \frac{\mathrm{Ra}_d}{\mathrm{GrSc}}\,y, \\
\gamma = \sqrt[4]{\frac{\mathrm{Ra}\,(\varepsilon+1) + \mathrm{Ra}_d}{4}}, \qquad S = \mathrm{Ra}\,(\varepsilon+1)(\cosh 2\gamma - \cos 2\gamma) + \gamma\,\mathrm{Ra}_d\,(\sin 2\gamma + \sinh 2\gamma).
\end{gathered} \tag{24}$$

Case 3: $\mathrm{Ra}\,(\varepsilon+1) + \mathrm{Ra}_d = 0$. The solution of the constitutive system is represented as

$$\begin{gathered}
u = 0, \qquad v = \frac{15(\varepsilon+1)}{2\,\mathrm{Ra}\,(\varepsilon+1) - 90}\,(x^3 - x), \qquad p = p_0 + \frac{\mathrm{Ra}}{2\mathrm{GrPr}}\Big(1 - \frac{\mathrm{Pr}}{\mathrm{Sc}}\,(\varepsilon+1)\Big)y^2, \\
T = \frac{\mathrm{Ra}\,(\varepsilon+1)}{8\,\mathrm{Ra}\,(\varepsilon+1) - 360}\,(3x^5 - 10x^3 + 15x) - \frac{45}{\mathrm{Ra}\,(\varepsilon+1) - 45}\,x + \frac{\mathrm{Ra}}{\mathrm{GrPr}}\,y, \\
C = -\frac{\mathrm{Ra}\,(\varepsilon+1)}{8\,\mathrm{Ra}\,(\varepsilon+1) - 360}\,(3x^5 - 10x^3 + 15x) - \frac{45\varepsilon}{\mathrm{Ra}\,(\varepsilon+1) - 45}\,x - \frac{\mathrm{Ra}\,(\varepsilon+1)}{\mathrm{GrSc}}\,y.
\end{gathered} \tag{25}$$

From the above formulas one can see that for $\varepsilon = -1$, the velocity vanishes and the temperature and concentration distributions become linear on the x coordinate. Therefore, mechanical equilibrium can occur in the system. In addition, in the first and second cases, it is possible to choose temperature and concentration gradients (or the corresponding Rayleigh numbers Ra and Ra_d) such that the pressure in the layer is constant with accuracy to the hydrostatic pressure.

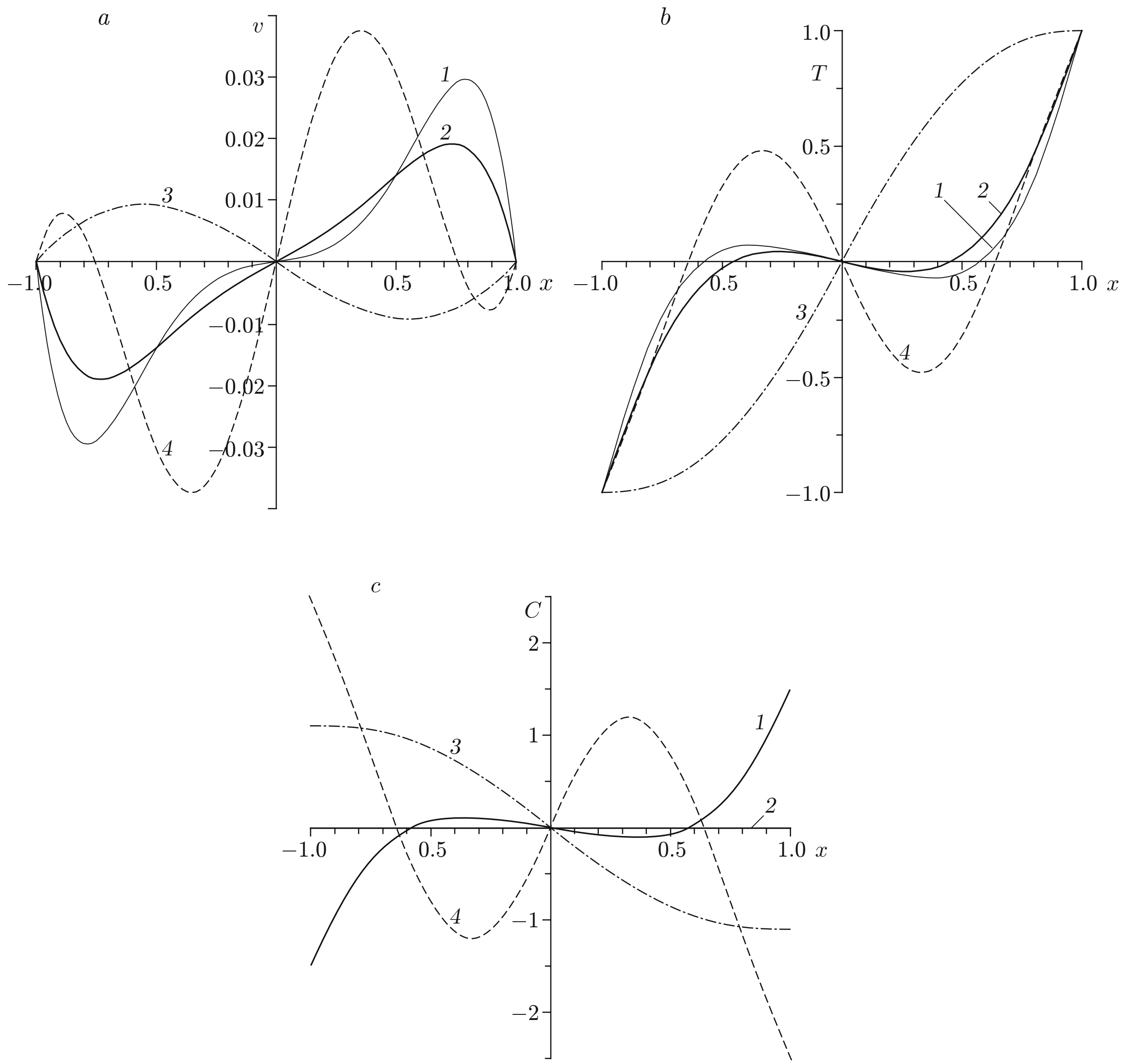

Fig. 4. Velocity (a), temperature (b), and concentration (c) profiles in the vertical layer for various values of the thermal-diffusion parameter: $\varepsilon = 1.5$ (1), 0 (2), -1.1 (3), and -2.5 (4).

The velocity, temperature, and concentration profiles in the cross section $y = 0$ for various values of the thermal-diffusion parameter are given in Fig. 4. These profiles correspond to the Rayleigh numbers $\mathrm{Ra} = 300$ and $\mathrm{Ra}_d = 0$. The functions v, T, and C for $y = 0$ are uniquely determined by specifying the indicated parameters.

In the absence of thermal diffusion (curve 2), the liquid rises up at the heated boundary and goes down at the cold boundary. In this case, the concentration ($C = 0$) is homogeneous. For $\varepsilon > 0$, normal thermal diffusion occurs and the light component diffuses toward the heated boundary. This results in an increase in the velocity (curve 1). For negative values of the parameter, anomalous thermal diffusion takes place. The light component moves toward the cold boundary, as a result of which the velocity of motion decreases. For $\varepsilon = -1$, mechanical equilibrium sets in. A further decrease in the thermal-diffusion parameter leads to inversion of the velocity profile (curve 3). The concentration of the light component at the cold boundary becomes high enough, so that near this boundary the liquid begins to rise up, and near the heated boundary, it goes down. As the parameter decreases further, inversion of the velocity profile is observed again (curve 4). Directly near the cold and heated boundaries, the liquid moves up and down, respectively, and in the middle of the layer, the direction of its motion becomes opposite. In this case, considerable temperature and concentration inhomegeneities in the layer are observed.

Solutions (23)–(25) extend the well-known solutions of the convection equations for a homogeneous liquid [1] and for a binary mixture [2–5] to the case of thermal diffusion of a mixture under various boundary conditions (the presence or absence of longitudinal temperature or concentration gradients and their various directions).

Conclusions. The group properties of the thermal-diffusion equations for a binary mixture are studied in the plane case. The admissible Lie operator algebra and the corresponding transformation group, which is infinite-dimensional, were found. The optimal first]- and second-order systems of subalgebras were constructed (the invariant solutions are classified). Examples of the exact solutions invariant with respect to the two-dimensional subalgebras from the optimal system were given. The search for such solutions reduces to integrating systems of ordinary differential equations. A physical interpretation of the results is proposed. The exact solutions describing thermal diffusion in an inclined layer with a free boundary and in a vertical layer in the presence of longitudinal temperature and concentration gradients were found. The effect of thermal diffusion on the flow regime was investigated.

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REFERENCES

1. G. Z. Gershuni, E. M. Zhukhovitskii, and A. A. Nepomnyashchii, *Stability of Convective Flows* [in Russian], Nauka, Moscow (1989).
2. G. Z. Gershuni, E. M. Zhukhovitskii, and L. E. Sorokin, "On the stability of plane parallel convective flow of a binary mixture," *Prikl. Mat. Mekh.*, **44**, No. 5, 823–830 (1980).
3. S. Yanase and K. Kohno, "The effect of a salinity gradient on the instability of natural convection in a vertical fluid layer," *J. Phys. Soc. Jpn.*, **54**, No. 10, 3747–3756 (1985).
4. G. Z. Gershuni, E. M. Zhukhovitskii, and L. E. Sorokin, "Stability of convective flow of a binary mixture with thermal diffusion," *Prikl. Mat. Mekh.*, **46**, No. 1, 66–71 (1982).
5. B. I. Nikolaev and A. A. Tubin, "Stability of convective flow of a binary mixture in a plane thermal-diffusion column," *Prikl. Mat. Mekh.*, **35**, No. 2, 248–254 (1971).
6. B. L. Smorodin, "Convection of a binary mixture under conditions of thermal diffusion and a variable temperature gradient," *J. Appl. Mech. Tech. Phys.*, **43**, No. 2, 217–223 (2002).
7. V. K. Andreev, "On the invariant solutions of the thermal-diffusion equations," in: *Proc. of the 3rd Int. Conf. on Symmetry and Differential Equations* (Krasnoyarsk, August 25–29, 2002), Krasnoyarsk State Univ., Krasnoyarsk (2002), pp. 13–17.
8. I. I. Ryzhkov "Optimal system of subalgebras for the thermal-diffusion equations," *Vychisl. Tekhnol.*, **9**, No. 1, 95–104 (2004).
9. L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).
10. L. V. Ovsyannikov, "Optimal systems of subalgebras," *Dokl. Ross. Akad. Nauk*, **333**, No. 6, 702–704 (1993).
11. L. V. Ovsyannikov, "The SUBMODEL program. Gas dynamics," *Prikl. Mat. Mekh.*, **58**, No. 4, 30–55 (1994).
12. V. F. Zaitsev and A. D. Polyanin, *Handbook on Ordinary Differential Equations* [in Russian], Fizmatlit, Moscow (2001).